CONNECTIVITY PROPERTIES OF MESH AND RING/MESH NETWORKS

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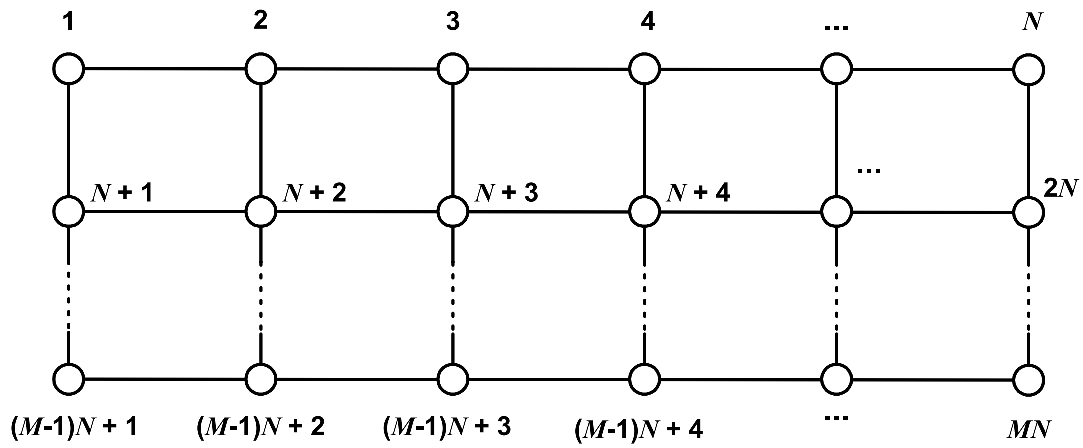
1. MESH NETWORK

A mesh network of NM nodes connected by bidirectional links can be modeled as M rows of N nodes, as illustrated in the following figure:



The adjacency (one-hop connectivity) matrix for such a network, in which a 1 entry at (i, j) indicates a connection from node i to node j and a 0 entry at $(i,\ j)$ indicates no connection from node i to node j, is an $NM \times NM$ matrix with the form given by

$$A_{M\times N} = \begin{bmatrix} A_N & I_N & 0 & \cdots & 0 \\ I_N & A_N & I_N & \cdots & 0 \\ 0 & I_N & A_N & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_N \end{bmatrix} \tag{1}$$

where I_N is the $N \times N$ identity matrix and A_N is the $N \times N$ adjacency matrix for a single row of N nodes connected in tandem. The structure of A_N is easily determined to be a matrix of 0s, except for $N-1$ 1s on the first upper diagonal and $N-1$ 1s on the first lower diagonal, for example,

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{2}$$

The one-hop connectivity of the network, defined as the fraction of the $NM(NM-1)$ possible links that are operative, is simply the sum of the elements of $A_{M\times N}$ divided by $NM(NM-1)$, or

$$\textit{Connectivity} = \frac{M\times 2(N-1)+2(M-1)\times N}{NM(NM-1)} = \frac{2[M(2N-1)-N]}{NM(NM-1)} \tag{3}$$

In addition to one-hop connectivity, we are interested in the hop distance between each pair of nodes, defined as the minimum number of links needed to be traversed in order to connect the pair. The hop distances for the possible node pairs can be represented collectively as entries in a (multihop) connectivity matrix. With some observation, it can easily be verified that the (multihop) connectivity matrix for the mesh network with M rows of N nodes has the form given by

$$C_{M\times N} = \begin{bmatrix} C_N & C_N+U_N & C_N+2U_N & \cdots & C_N+(M-1)U_N \\ C_N+U_N & C_N & C_N+U_N & \cdots & C_N+(M-2)U_N \\ C_N+2U_N & C_N+U_N & C_N & \cdots & C_N+(M-3)U_N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_N+(M-1)U_N & C_N+(M-2)U_N & C_N+(M-3)U_N & \cdots & C_N \end{bmatrix} \tag{4}$$

where we use U_N to denote an $N\times N$ matrix of all 1s, and C_N is the connectivity matrix for a row of N nodes, given by

$$C_N = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & \cdots & N-1 \\ 1 & 0 & 1 & 2 & 3 & \cdots & N-2 \\ 2 & 1 & 0 & 1 & 2 & \cdots & N-3 \\ 3 & 2 & 1 & 0 & 1 & \cdots & N-4 \\ 4 & 3 & 2 & 1 & 0 & \cdots & N-5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ N-1 & N-2 & N-3 & N-4 & N-5 & \cdots & 0 \end{bmatrix} \tag{5}$$

Note that the elements on the kth upper and lower diagonals are all equal to k, $k = 1, 2, \ldots, N-1$. For example, the connectivity matrix for a mesh network with three rows of four nodes is the following 12×12 matrix:

$$C_{3\times4} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 2 & 3 & 4 & 2 & 3 & 4 & 5 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 & 1 & 2 & 3 & 3 & 2 & 3 & 4 \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 & 2 & 1 & 2 & 4 & 3 & 2 & 3 \\ \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 \\ 1 & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & 2 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} & 3 & 2 & 1 & 2 \\ 4 & 3 & 2 & 1 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ 3 & 2 & 3 & 4 & 2 & 1 & 2 & 3 & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} \\ 4 & 3 & 2 & 3 & 3 & 2 & 1 & 2 & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 5 & 4 & 3 & 2 & 4 & 3 & 2 & 1 & \mathbf{3} & \mathbf{2} & \mathbf{1} & \mathbf{0} \end{bmatrix} \tag{6}$$

The maximum hop distance between node pairs is $N + M - 2$. The average hop distance is simply the sum of the elements of $C_{M\times N}$ divided by $NM(NM-1)$. To derive the average hop distance for the mesh network, let us use the notation $\|X\|_+$ to denote the sum of all the elements of matrix X. Then the average hop distance is given by

$$\overline{m} = \frac{\|C_{M\times N}\|_+}{NM(NM-1)} \tag{7}$$

By inspection of (4), the sum of the elements of $C_{M\times N}$ is given by

$$\begin{aligned} \|C_{M\times N}\|_+ &= M^2\|C_N\|_+ + 2(M-1)\|U_N\|_+ + 2(M-2)\|2U_N\|_+ \\ &\qquad + \cdots + 2(2)\|(M-2)U_N\|_+ + 2(1)\|(M-1)U_N\|_+ \\ &= M^2\|C_N\|_+ + 2\|U_N\|_+[(M-1)\cdot 1 + (M-2)\cdot 2 \\ &\qquad + \cdots + 2\cdot(M-2) + 1\cdot(M-1)] \end{aligned} \tag{8}$$

where $\|U_N\|_+ = N^2$ and

$$\begin{aligned} \|C_N\|_+ &= 2[(N-1)\cdot 1 + (N-2)\cdot 2 + \cdots + 2\cdot(N-2) + 1\cdot(N-1)] \\ &= 2\sum_{k=1}^{N-1}(N-k)k = 2N\sum_{k=1}^{N-1}k - 2\sum_{k=1}^{N-1}k^2 \\ &= 2N\cdot\tfrac{1}{2}(N-1)N - 2\cdot\tfrac{1}{6}(N-1)N(2N-1) = N(N-1)\left(\tfrac{N+1}{3}\right) \end{aligned} \tag{9}$$

The coefficient of $\|U_N\|_+$ is the same as (9), but with M exchanged for N; thus

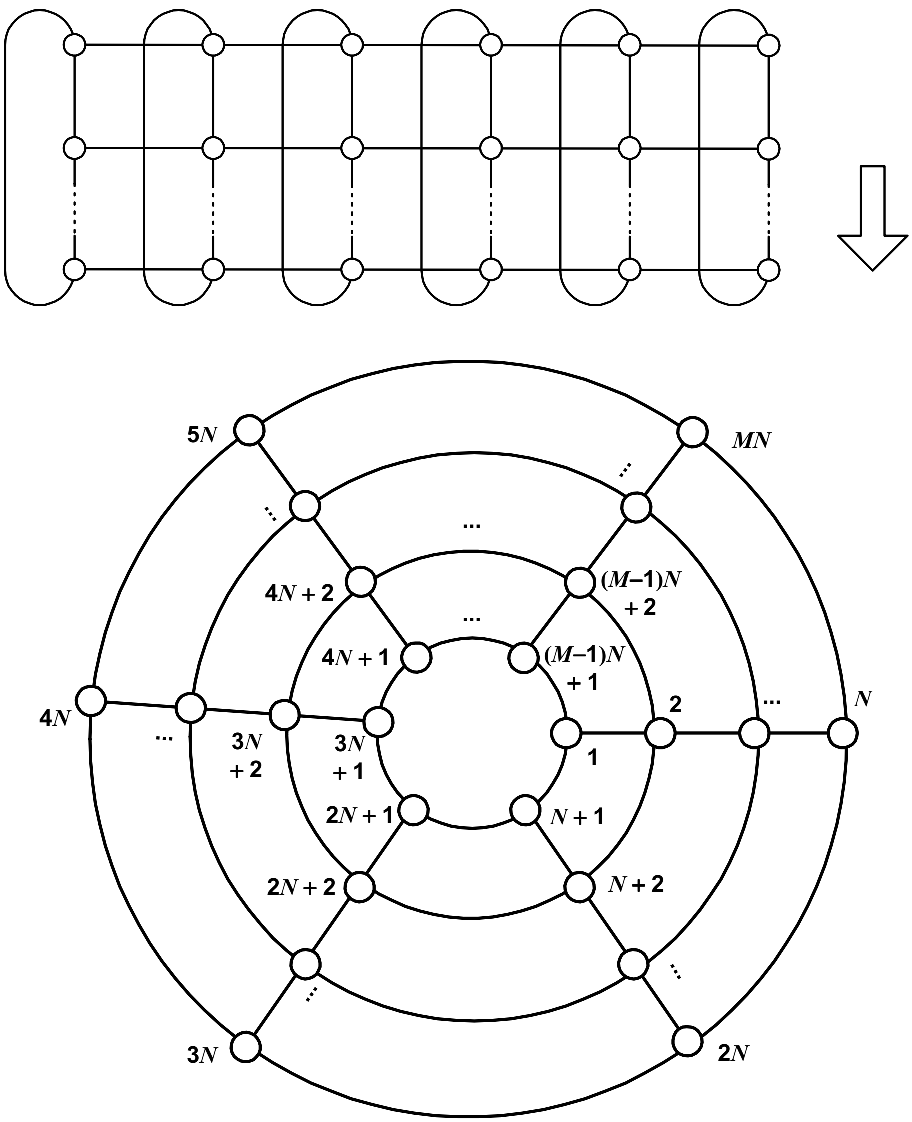
$$\begin{aligned} \|C_{M\times N}\|_+ &= M^2\cdot N(N-1)\left(\tfrac{N+1}{3}\right) + N^2\cdot M(M-1)\left(\tfrac{M+1}{3}\right) \\ &= \tfrac{1}{3}NM[M(N-1)(N+1) + N(M-1)(M+1)] \\ &= \tfrac{1}{3}NM[M(N^2-1) + N(M^2-1)] \\ &= \tfrac{1}{3}NM[NM(N+M) - (N+M)] = NM(NM-1)\left(\tfrac{N+M}{3}\right) \end{aligned} \tag{10}$$

and the average hop distance for a mesh network is

$$\overline{m} = \frac{N + M}{3} \tag{11}$$

2. RING-MESH NETWORK

As indicated by the following figure, the network connection pattern that we call "ring-mesh" can be viewed as a redrawing of a modified mesh network with M rows of N nodes in which the "top" nodes in each of the N columns is given a connection to the "bottom" node in the column, thereby forming N rings each containing M nodes with M radial connections from inner ring to outer ring.



The adjacency (one-hop connectivity) matrix for such a network, in which a 1 entry at (i, j) indicates a connection from node i to node j and a 0 entry at (i, j) indicates no connection from node i to node j, is an $NM \times NM$ matrix with the form given by

$$A_{\textit{ring-mesh}} = \begin{bmatrix} A_N & I_N & 0 & \cdots & I_N \\ I_N & A_N & I_N & \cdots & 0 \\ 0 & I_N & A_N & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_N & 0 & 0 & \cdots & A_N \end{bmatrix} \tag{12}$$

where I_N is the $N \times N$ identity matrix and A_N is the $N \times N$ adjacency matrix for a single row of N nodes connected in tandem. The matrix structure of (12) differs from that of (1) only in that the first and last of the M rows have two identity matrices instead of just one. The structure of A_N is a matrix of 0s, except for $N-1$ 1s on the first upper diagonal and $N-1$ 1s on the first lower diagonal as shown by example in (2). The one-hop connectivity of the network, defined as the fraction of the $NM(NM-1)$ possible links that are operative, is simply the sum of the elements of $A_{\textit{ring-mesh}}$ divided by $NM(NM-1)$, or

$$\textit{Connectivity} = \frac{M \times 2(N-1) + 2M \times N}{NM(NM-1)} = \frac{2(2N-1)}{N(NM-1)} \tag{13}$$

With some observation, it can be verified that the (multihop) connectivity matrix for the ring-mesh network with N rings of M nodes has the form given by

$$C_{\textit{ring-mesh}} = \begin{bmatrix} C_N & C_N+U_N & C_N+2U_N & \cdots & C_N+2U_N & C_N+U_N \\ C_N+U_N & C_N & C_N+U_N & \cdots & C_N+3U_N & C_N+2U_N \\ C_N+2U_N & C_N+U_N & C_N & \cdots & C_N+4U_N & C_N+3U_N \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ C_N+2U_N & C_N+3U_N & C_N+4U_N & \cdots & C_N & C_N+U_N \\ C_N+U_N & C_N+2U_N & C_N+3U_N & \cdots & C_N+U_N & C_N \end{bmatrix} \tag{14}$$

where we use U_N to denote an $N \times N$ matrix of all 1s, and C_N is the connectivity matrix for a row of N nodes, given by (5). In words, the structure represented in (14) is an $M \times M$ array of $N \times N$ matrices in which the main diagonal matrices all equal C_N, while the first upper and first lower diagonals and the upper right and lower left corners all equal $C_N + U_N$; the diagonals in between equal C_N plus an ascending and descending multiple of U_N. The multiple of U_N on the upper and lower diagonals is the series $\left\{1,\, 2,\, 3,\, \ldots,\, \tfrac{1}{2}(M-1),\, \tfrac{1}{2}(M-1),\, \ldots,\, 3,\, 2,\, 1\right\}$ for odd values of M and is the series $\left\{1,\, 2,\, 3,\, \ldots,\, \tfrac{1}{2}M-1,\, \tfrac{1}{2}M,\, \tfrac{1}{2}M-1,\, \ldots,\, 3,\, 2,\, 1\right\}$ for even values of M.

For example, the connectivity matrix for a ring mesh network with $N = 3$ rings of $M = 4$ nodes is the following 12×12 matrix:

$$C_{\textit{ring-mesh}} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{2} & 1 & 2 & 3 & 2 & 3 & 4 & 1 & 2 & 3 \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 \\ \mathbf{2} & \mathbf{1} & \mathbf{0} & 3 & 2 & 1 & 4 & 3 & 2 & 3 & 2 & 1 \\ 1 & 2 & 3 & \mathbf{0} & \mathbf{1} & \mathbf{2} & 1 & 2 & 3 & 2 & 3 & 4 \\ 2 & 1 & 2 & \mathbf{1} & \mathbf{0} & 1 & 2 & 1 & 2 & 3 & 2 & 3 \\ 3 & 2 & 1 & \mathbf{2} & \mathbf{1} & \mathbf{0} & 3 & 2 & 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 & 2 & 3 & \mathbf{0} & 1 & \mathbf{2} & 1 & 2 & 3 \\ 3 & 2 & 3 & 2 & 1 & 2 & 1 & \mathbf{0} & \mathbf{1} & 2 & 1 & 2 \\ 4 & 3 & 2 & 3 & 2 & 1 & \mathbf{2} & \mathbf{1} & \mathbf{0} & 3 & 2 & 1 \\ 1 & 2 & 3 & 2 & 3 & 4 & 1 & 2 & 3 & \mathbf{0} & \mathbf{1} & \mathbf{2} \\ 2 & 1 & 2 & 3 & 2 & 3 & 2 & 1 & 2 & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ 3 & 2 & 1 & 4 & 3 & 2 & 3 & 2 & 1 & \mathbf{2} & \mathbf{1} & \mathbf{0} \end{bmatrix} \tag{15}$$

The maximum hop distance between node pairs is $N-1+\frac{1}{2}(M-1)$ for M odd and $N-1+\frac{1}{2}M$ for M even. The average hop distance is simply the sum of the elements of $C_{\textit{ring-mesh}}$ divided by $NM(NM-1)$. To derive the average hop distance for the mesh network, we use the notation $\|X\|_+$ to denote the sum of all the elements of matrix X. Then the average hop distance is given by

$$\overline{m} = \frac{\|C_{\textit{ring-mesh}}\|_+}{NM(NM-1)} \tag{16}$$

By inspection of (14), the sum of the elements of $C_{\textit{ring-mesh}}$ is given by

$$\|C_{\textit{ring-mesh}}\|_+ = M^2\|C_N\|_+ + M\|U_N\|_+ S \tag{17}$$

where $\|U_N\|_+ = N^2$, $\|C_N\|_+ = N(N-1)\left(\frac{N+1}{3}\right)$, and

$$S = \begin{cases} 1+2+\cdots+\frac{M-1}{2}+\frac{M-1}{2}+\cdots+2+1, & M \text{ odd} \\ 1+2+\cdots+\frac{M}{2}-1+\frac{M}{2}+\frac{M}{2}-1+\cdots+2+1, & M \text{ even} \end{cases} \tag{18a}$$

$$= \begin{cases} \frac{1}{4}(M^2-1), & M \text{ odd} \\ \frac{1}{4}M^2, & M \text{ even} \end{cases} \tag{18b}$$

Thus

$$\begin{aligned} \|C_{\textit{ring-mesh}}\|_+ &= M^2 \cdot N(N-1)\left(\tfrac{N+1}{3}\right) + M \cdot N^2 S \\ &= \frac{NM}{12} \times \begin{cases} 4M(N^2-1)+3N(M^2-1), & M \text{ odd} \\ 4M(N^2-1)+3NM^2, & M \text{ even} \end{cases} \end{aligned} \tag{19}$$

and the average hop distance for a ring-mesh network is

$$\overline{m} = \frac{1}{12(NM-1)} \times \begin{cases} 4M(N^2-1)+3N(M^2-1), & M \text{ odd} \\ 4M(N^2-1)+3NM^2, & M \text{ even} \end{cases} \tag{20}$$